THE MOTION OF A BODY WITH A CAVITY PARTLY FILLED WITH A VISCOUS LIQUID

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Many papers are concerned with the dynamics of a rigid body with a cavity filled with liquid (see the bibliography in [1]). The present paper deals with the motion of a rigid body having a cavity partly filled with a viscous incompressible liquid, and having a free surface. The shape of the cavity is arbitrary. The problem is considered in a linear formulation. The oscillations of the body with respect to its center of inertia and the motion of the liquid in the cavity are assumed small. The viscosity of the liquid is considered low. The solution of the problem of the oscillations of a body with a cavity partly filled with an ideal liquid is used as an initial approximation [1 to 6]. The viscosity is taken into consideration by the boundary layer method used before in similar problems [1 and 7 to 10). General equations are derived for the dynamics of a body filled with a liquid, for an arbitrary form of cavity. The coefficients of those integro-differential equations depend only on the solution of the problem of the oscillations of a body with a cavity of the given form filled with an ideal liquid. Since the corresponding problem has been solved for cavities of many forms [1 to 6, 11 and 12] in the case of an ideal liquid, the determination of the characteristic coefficients is reduced to the evaluation of quadratures. Several particular cases of motion are considered.

1. Statement of the problem. Let a rigid body of mass m_1 have a cavity containing a mass m_2 of a viscous incompressible liquid of density ρ and kinematic viscosity ν and also a gas at a constant pressure p_0. We shall neglect the influence of the motion of the gas.

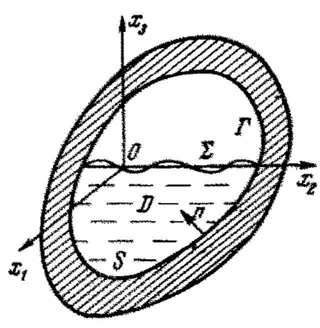

Fig. 1

For an unperturbed motion we shall consider the translation of the body together with the liquid. The liquid fills the domain D bounded by the surface of the cavity walls S and the surface Σ. Let us take for reference with respect to the rigid body an arbitrary point O of the surface Σ. Let us introduce two cartesian systems of coordinates, $Ox_1x_2x_3$

fixed with respect to the body, and $Oy_1y_2y_3$ which moves with the point O. We shall assume that in the unperturbed motion those systems of coordinates coincide, that the surface Σ lies in the plane Ox_1x_2 and that in the domain D occupied by the liquid we have $x_3 \leq 0$ (Fig.1).

In the system of coordinates $Oy_1y_2y_3$ the body and the liquid are subject to forces due to the masses; their acceleration $\mathbf{g} = \mathbf{g}_0 - \mathbf{R}_0^{\cdot\cdot}$ results from the acceleration due to the gravity $\mathbf{g}_0$ and the forces of inertia. Here $\mathbf{R}_0$ is the absolute radius vector of the point O and the dots refer to the derivatives with respect to time t. In the unperturbed case, the acceleration of the masses $\mathbf{g}$ has a constant direction opposite to that of the axis x_3. Since $\mathbf{g}_0$ is a constant vector, this is possible in the following cases: (1) the acceleration of $\mathbf{R}_0^{\cdot\cdot}$ of the translation motion is constant in magnitude and direction; (2) the acceleration $\mathbf{R}_0^{\cdot\cdot}$ is colinear with the vector $\mathbf{g}_0$ and varies arbitrarily in magnitude; (3) the acceleration $\mathbf{R}_0^{\cdot\cdot}$ is constant in direction and $\mathbf{g}_0 = 0$ (case of weightlessness).

Looking now at the perturbed motion, we shall consider that the angles between the coordinate axes $Ox_1x_2x_3$ and $Oy_1y_2y_3$, the absolute angular velocity $\boldsymbol{\omega}$ of the body, and the projections of the acceleration $\mathbf{g}$ on the axes x_1 and x_2 are small quantities of the first order. The velocity $\mathbf{u}$ of the liquid in the system of coordinates $Oy_1y_2y_3$ and function f determining the equation of the disturbed free surface of the liquid $x_3 = f(x_1, x_2, t)$, have the same order of magnitude. The problem is considered in the linear formulation. Let us write the momentum equation and the equation of the moments for the system "body + liquid"

$$\mathbf{Q}^{\cdot} = \mathbf{F} + m\mathbf{g}, \qquad \mathbf{K}^{\cdot} = \mathbf{M} + (m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \times \mathbf{g}, \qquad m = m_1 + m_2$$

$$\mathbf{Q} = m_1\mathbf{r}_1^{\cdot} + m_2\mathbf{r}_2^{\cdot}, \qquad \mathbf{K} = \mathbf{J}_1 \cdot \boldsymbol{\omega} + \rho \int_D \mathbf{r} \times \mathbf{u}\, dV \qquad (1.1)$$

Here $\mathbf{Q}$ is the momentum of the body with liquid in the system $Oy_1y_2y_3$, $\mathbf{K}$ is the moment, with respect to O, of the momentum in the same system, $\mathbf{F}$ is the main vector of all external forces applied to the body (except for the forces of gravity and inertia), $\mathbf{M}$ is the main moment of those forces with respect to the reference O. The vector-radii of the centers of inertia of the body and of the liquid are represented by $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively (those vectors, as well as $\mathbf{r}$, are measured from the origin O), and $\mathbf{J}_1$ is the tensor of inertia of the body with respect to the point O.

With an accuracy up to the higher order terms we have (1.2)

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = m_1\mathbf{r}_1 + \rho \int_D \mathbf{r}\, dV + \rho \int_\Sigma \mathbf{r} f\, dS = m\mathbf{r}_c + \rho\left(\mathbf{e}_1 \int_\Sigma x_1 f\, dS + e_2 \int_\Sigma x_2 f\, dS\right)$$

Here $\mathbf{r}_c$ is the vector radius (with respect to the reference O) of the center of inertia of the system "body + liquid" in the unperturbed state (when the liquid is at rest in the system of coordinates $Ox_1x_2x_3$), $\mathbf{e}_i$ are the unit vectors of the axes Ox_i, $i = 1, 2, 3$. It is evident that the vector $\mathbf{r}_c$, as well as the vector $\mathbf{r}_1$, are stationary in the system $Ox_1x_2x_3$.

The derivatives of the unit vectors $\mathbf{e}_i$ are determined by Equations

$$\mathbf{e}_i^{\cdot} = \boldsymbol{\omega} \times \mathbf{e}_i \qquad (i = 1, 2, 3) \tag{1.3}$$

and are small quantities of the first order. We note also that according to the assumptions made, we have an accuracy up to the higher order terms

$$\mathbf{g} = -g\mathbf{e}_3 + g_1\mathbf{e}_1 + g_2\mathbf{e}_2, \quad g_i = \mathbf{g}\cdot\mathbf{e}_i, \quad |g_i| \ll g = |\mathbf{g}| \qquad (i = 1, 2, 3) \tag{1.4}$$

Substituting (1.2) to (1.4) into Equations (1.1) and neglecting higher order terms, we have

$$\mathbf{Q}^{\cdot} = \mathbf{F} + m\mathbf{g}, \qquad \mathbf{g} = \mathbf{g}_0 - \mathbf{R}_0^{\cdot\cdot}, \qquad m = m_1 + m_2$$

$$\mathbf{K}^{\cdot} = \mathbf{M} + m\dot{\mathbf{r}}_c \times \mathbf{g} + \rho g\Big(\mathbf{e}_2 \int_\Sigma x_1 f dS - \mathbf{e}_1 \int_\Sigma x_2 f dS\Big) \tag{1.5}$$

$$\mathbf{Q} = m\boldsymbol{\omega} \times \mathbf{r}_c + \rho\Big(\mathbf{e}_1 \int_\Sigma x_1 \frac{\partial f}{\partial t}\, dS + \mathbf{e}_2 \int_\Sigma x_2 \frac{\partial f}{\partial t}\, dS\Big) \qquad \mathbf{K} = \mathbf{J}_1\cdot\boldsymbol{\omega} + \rho \int_D \mathbf{r} \times \mathbf{u} dV$$

The linearized equations of motion of the liquid with respect to the system $Oy_1y_2y_3$ and the boundary conditions have the form

$$\mathbf{u}_t = -\rho^{-1}\nabla p' + \mathbf{g} + \nu\Delta\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } D, \quad \mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{on } S$$

$$\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \qquad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0, \qquad p' - 2\rho\nu\frac{\partial u_3}{\partial x_3} = p_0$$

$$u_3 = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{e}_3 + \partial f / \partial t \qquad \text{for } x_3 = f(x_1, x_2, t) \tag{1.6}$$

Here the index t denotes a partial derivative, p' is the pressure in the liquid, the u_i are the projections on the axes x_i of the velocity $\mathbf{u}$ of the liquid with respect to the system $Oy_1y_2y_3$. The last equation of (1.6) expresses the kinematic condition on the free surface. Let us introduce a new unknown function

$$p = (p' - p_0) / \rho - \mathbf{g}\cdot\mathbf{r}$$

and let us rewrite the relations (1.6) taking (1.4) into account and expressing the conditions on the nonperturbed free surface Σ

$$\mathbf{u}_t = -\nabla p + \nu\Delta\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } D, \qquad \mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{on } S$$

$$u_3 = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{e}_3 + \partial f / \partial t, \qquad p - gf + g_1x_1 + g_2x_2 = 2\nu\partial u_3 / \partial x_3$$

$$\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \quad \text{on } \Sigma \tag{1.7}$$

Equations (1.5) and (1.7) describe the dynamics of the body with the liquid. It is necessary to add to them the usual kinematic relations (for instance in the form (1.3)), and also possibly, the other equations which complete the system; for instance, the equations of the control system, the equations concerning the other solids, coupled with the body under consideration, and so on. When considering Cauchy's problem, it is necessary to specify the initial conditions for the body (position and velocity of the body) and the liquid

$$\mathbf{u}(\mathbf{r}, t_0) = \mathbf{u}_0(\mathbf{r}), \qquad f(x_1, x_2, t_0) = f_0(x_1, x_2) \tag{1.8}$$

The function $\mathbf{u}_0$ must satisfy the continuity condition, and also, together with f_0, the boundary conditions (1.7) at the instant t_0 .

Below, Equations (1.5) and (1.7) are investigated in the case of a high Reynolds number: $l^2T^{-1}\nu^{-1} \gg 1$, i.e. for the case of a low viscosity. Here l is the characteristic linear dimension of the cavity, T is the characteristic time, the order of magnitude of which is that of the period of the oscillations of the body or of the liquid inside the cavity. Without loss of generality, choosing l and T for units of length and time we get $\nu \ll 1$. Thus the problem reduces to the asymptotic solution of the equations of hydrodynamics (1.7) for $\nu \ll 1$ and the ensuing simplification of Equations (1.5).

2. Analysis of the hydrodynamics equations. We seek the solution of the problem (1.7) as was done in [9 and 10] by the method of the boundary layer [13], assuming

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \qquad p = q + s, \qquad \mathbf{v} = \mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1 + \ldots \tag{2.1}$$

$$q = q^\circ + \nu^{1/2}q^1 + \ldots, \qquad f = f^\circ + \nu^{1/2}f^1 + \ldots \qquad (\nu \ll 1)$$

Here the superscript refers to the number of the approximation. The functions $\mathbf{w}$ and s are functions of the boundary layer type. They can also be expanded in series of the powers of $\nu^{1/2}$, whereupon all the coefficients of the terms in $\mathbf{w}^k$ and s^k decrease rapidly when the distance to the boundaries of the domain D increases. Let us denote by D_S and D_Σ the domains of the boundary layer adjacent from the inside to the surfaces S and Σ, and having a thickness of the order of $\nu^{1/2}$. Then outside D_S and D_Σ we can assume $\mathbf{w} = 0$ and $s = 0$.

The functions $\mathbf{w}$ and s satisfy Equations (1.7) in the same manner as the functions $\mathbf{v}$ and q. Their boundary conditions are obtained by the following recurrent process. Let us assume the functions $\mathbf{v}^i$, q^i, $\mathbf{w}^i$, s^i and f^i have already been found for $i = 0, 1, .., k-1$. To determine the functions $\mathbf{v}^k$, q^k and f^k we shall require that they satisfy, together with the approximations found earlier, the condition $\mathbf{u}\cdot\mathbf{n} = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{n}$ on the wall S and the first two conditions (1.7) on Σ. Here $\mathbf{n}$ is the unit vector of the normal to S directed inwards. Then we determine the functions $\mathbf{w}^k$ and s^k which, as the functions found earlier, must satisfy the condition $\mathbf{u}^* = (\boldsymbol{\omega} \times \mathbf{r})^*$ on S, the last two conditions (1.7) on Σ and also the conditions $\mathbf{w}^k \to 0$ and $s^k \to 0$ outside D_S and D_Σ. The asterisk refers always to the projection of the vector on the plane tangent to the surface S. The recurrent process described leads to the series (2.1) each subsequent term of which is a quantity of order $\nu^{1/2}$ with respect to the previous one.

For the functions $\mathbf{v}^k$ and q^k which satisfy Equations (1.7) we have

$$\mathbf{v}_t^k = -\nabla q^k + \Delta\mathbf{v}^{k-2}, \ \operatorname{div} \mathbf{v}^k = 0 \qquad (k = 0, 1, \ldots) \tag{2.2}$$

whereby for $k = 0, 1$ the term $\Delta\mathbf{v}^{k-2}$ does not appear in the first equation of (2.2).

Thus we have $(\operatorname{rot} \mathbf{v}^\circ)_t = (\operatorname{rot} \mathbf{v}^1)_t = 0$. We shall assume that the initial distribution of the velocities (1.8) is potential everywhere, except, maybe on the boundary layer domains D_S, D_Σ. Then at $t = t_0$, we shall have $\operatorname{rot} \mathbf{v}^k = 0$ in D for all functions $\mathbf{v}^k$. Consequently, $\operatorname{rot} \mathbf{v}^\circ = \operatorname{rot} \mathbf{v}^1 = 0$ in D for all $t \geq t_0$. By induction it can be easily proved that rot $\mathbf{v}^k = 0$, $\Delta\mathbf{v}^k = 0$ for all k when $t \geq t_0$. Without loss of generality and taking (2.2) into consideration, we shall assume that

$$\mathbf{v}^k = \nabla\varphi^k, \qquad q^k = -\varphi_t^k, \qquad \Delta\varphi^k = 0 \qquad (k = 0,1,\ldots) \tag{2.3}$$

Below we shall limit ourselves to the determination of the functions φ^k and f^k (and consequently $\mathbf{v}^k$ and q^k) for $k = 0, 1$ and also of the functions $\mathbf{w}^\circ$ and s° which will be simply denoted by $\mathbf{w}$ and s (i.e. only by those terms which are expressed in (2.1)).

The boundary condition for the functions φ° and f°, as it follows from (2.3) and (1.7) and the described method of construction of the solution in the form (2.1), will be as follows:

$$\Delta\varphi^\circ = 0 \text{ in } D, \qquad \partial\varphi^\circ / \partial n = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{n} \qquad \text{on } S$$

$$\partial\varphi^\circ / \partial x_3 = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{e}_3 + \partial f^\circ / \partial t, \quad \varphi_t^\circ + gf^\circ - g_1x_1 - g_2x_2 = 0 \quad \text{on } \Sigma \tag{2.4}$$

This is the usual problem for an ideal liquid, the solution of which as known from [1 and 3 to 5], can be sought in the form

$$\varphi^\circ = \omega_1\Phi_1 + \omega_2\Phi_2 + \omega_3\Phi_3 + \sum_{k=0}^{\infty} a_k{}^\circ(t)\,\psi_k(x_1, x_2, x_3)$$

$$f^\circ = \sum_{k=0}^{\infty} b_k{}^\circ(t)\,\psi_k(x_1, x_2, 0) \tag{2.5}$$

Here $\omega_i = \boldsymbol{\omega}\cdot\mathbf{e}_i$ represent the projections of the vector $\boldsymbol{\omega}$ on the axes x_i ; the $a_k{}^\circ$, $b_k{}^\circ$ are coefficients yet unknown and the Φ_i , ψ_k represent time independent functions of x_1 , x_2 and x_3 . The functions Φ_i are Zhukovskii potentials for the domain D^* , under the condition that the free surface Σ is replaced by a rigid wall. They satisfy the boundary conditions

$$\Delta\Phi_i = 0 \text{ in } D, \quad \partial\Phi_i / \partial n = (\mathbf{r} \times \mathbf{n})\cdot\mathbf{e}_i \quad \text{on } S$$

$$\partial\Phi_i / \partial x_3 = (\mathbf{r} \times \mathbf{e}_3)\cdot\mathbf{e}_i \quad \text{on } \Sigma \quad (i = 1, 2, 3) \tag{2.6}$$

and are determined with an accuracy up to any arbitrary constants. The functions ψ_k are the eigenfunctions of the problem of the free oscillations of an ideal liquid (2.7)

$$\Delta\psi_k = 0 \text{ in } D, \qquad \frac{\partial\psi_k}{\partial n} = 0 \quad \text{on } S, \qquad \frac{\partial\psi_k}{\partial x_3} = \lambda_k{}^2\psi_k \quad \text{on } \Sigma \qquad (k = 0, 1, \ldots)$$

It is known [1], that the problem (2.7) has an infinite discrete spectrum of finite eigenvalues λ_k , whereupon $\lambda_0 = 0$, and all other λ_k are real and positive. The functions $\psi_k(x_1, x_2, x_3)$ represent a complete orthogonal system in D and the functions $\psi_k(x_1, x_2, 0)$ a complete orthogonal system in Σ . Thus, using Green's theorem and the boundary conditions (2.7) we have

$$\int_\Sigma \psi_k\psi_l\,dS = \frac{1}{\lambda_k{}^2}\int_\Sigma \frac{\partial\psi_k}{\partial x_3}\psi_l\,dS = \frac{1}{\lambda_k{}^2}\int_\Sigma \frac{\partial\psi_l}{\partial x_3}\psi_k dS = \frac{\lambda_l{}^2}{\lambda_k{}^2}\int_\Sigma \psi_k\psi_l\,dS$$

There follows the orthogonality of the functions $\psi_k(x_1, x_2, 0)$ and $\psi_l(x_1, x_2, 0)$ on Σ for $\lambda_k \neq \lambda_l$. We shall subject the functions ψ_k to the norm condition

$$\int_\Sigma \psi_k\psi_l = \delta_{kl} \qquad (k, l = 0, 1, \ldots) \tag{2.8}$$

(where δ_{kl} is Kronecker's symbol).

By virtue of (2.6) and (2.7) the expansions (2.5) satisfy the equations and the condition (2.4) concerning S. In order to satisfy also the conditions (2.4) on the surface Σ, we shall substitute in those conditions the expansions (2.5) and we shall express the derivatives $\partial\Phi_i/\partial x_3$ and $\partial\psi_k/\partial x_3$ on Σ in terms of the conditions (2.6),(2.7). The equations obtained in the domain Σ (for $x_3 = 0$) will be expanded in Fourier series along the orthogonal system of functions $\psi_k(x_1, x_2, 0)$ and the coefficients corresponding to the same ψ_k equated to each other. We obtain a system of equations for the coefficients of the expansions (2.5). This system is given in [1, 3 and 4] in somewhat different notations. (2.9)

$$(b_k^\circ)^\cdot = \lambda_k^2 a_k^\circ, \qquad (a_k^\circ)^\cdot + g b_k^\circ - g_1 A_{1k} - g_2 A_{2k} + \sum_{i=1}^{3} \omega_i^\cdot B_{ik} = 0 \quad (k = 0, 1, \ldots)$$

The following notations were introduced for the Fourier coefficients:

$$A_{jk} = \int_\Sigma x_j \psi_k dS \quad (j = 1, 2), \quad B_{ik} = \int_\Sigma \Phi_i \psi_k dS \quad (i = 1, 2, 3; \quad k = 0, 1, \ldots) \quad (2.10)$$

Let us determine the functions $\mathbf{w}$ and s which we shall seek separately in the domains D_S and D_Σ (similar solutions can be found in [1, 9 and 10]). In the domain D_S, which has a thickness of the order of $\nu^{1/2}$ and is adjacent from the inside to the surface S, we shall introduce curvilinear orthogonal coordinates ξ, η and ζ such that the surface Σ is the surface $\zeta = 0$, whereby $\zeta > 0$, inside the domain D_S.

Let w_ξ, w_η and w_ζ be the components of the vector $\mathbf{w}$ in those coordinates, H_ξ, H_η and H_ζ the corresponding coefficients of Lamé, and H_ξ°, H_η°, H_ζ° the values of those coefficients for $\zeta = 0$. Without loss of generality, let us take $H_\zeta^\circ = 1$, then ζ is the distance to the surface S along the internal normal $\mathbf{n}$.

Let us write the equations of motion (1.7) substituting in them $\mathbf{u}$ for $\mathbf{w}$, p for s, in the coordinates ξ, η and ζ. Then let us make the change of variables

$$\zeta = \nu^{1/2}\alpha, \qquad w_\zeta = \nu^{1/2} w_\alpha \quad (2.11)$$

and take the limit when $\nu \to 0$ in the equations of motion.

The equation of motion corresponding to the coordinate α has the simple form $\partial s/\partial\alpha = 0$. Since outside the boundary layer (for $\alpha \to \infty$) we have $s \to 0$, then $s \equiv 0$ in D_S. Taking this relation into consideration, the remaining equations of motion and the equation giving the condition of continuity, have the form

$$\frac{\partial \mathbf{w}^*}{\partial t} = \frac{\partial^2 \mathbf{w}^*}{\partial \alpha^2}, \qquad \operatorname{Div} \mathbf{w}^* + \frac{\partial w_\alpha}{\partial \alpha} = 0$$

$$\operatorname{Div} \mathbf{w}^* = \frac{1}{H_\xi^\circ H_\eta^\circ}\left[\frac{\partial (H_\eta^\circ w_\xi)}{\partial \xi} + \frac{\partial (H_\xi^\circ w_\eta)}{\partial \eta}\right] \quad (2.12)$$

Here $\mathbf{w}^*$ is a two-dimensional vector (with components w_ξ and w_η) tangent to the surface S, Div represents the two-dimensional operation of divergence for the vector fields on the surface S. We shall also give, in agreement with the recurrent process of construction of the solution outlined above, the boundary conditions and also the initial condition

$$\mathbf{w}^* = \boldsymbol{\omega} \times \mathbf{r} - \nabla\varphi^\circ \quad \text{for } \alpha = 0, \qquad \mathbf{w}^* \to 0,\ w_\alpha \to 0 \quad \text{for } \alpha \to \infty$$
$$\mathbf{w}^* = (\mathbf{u}_0 - \nabla\varphi^\circ)^* \quad \text{for } t = t_0 \tag{2.13}$$

Let us point out that on the basis of the condition (2.4), the vector $\boldsymbol{\omega} \times \mathbf{r} - \nabla\varphi^\circ$, as well as $\mathbf{w}^*$, lies in the plane tangent to S. The initial distribution of the velocities $\mathbf{u}_0$ is assumed such that with the exception of the potential component $\nabla\varphi^\circ$, it is concentrated only in the boundary layer. The asterisk denotes its projection on the plane tangent to S; the third component w_α of the vector $\mathbf{w}^*$ is determined from the equations of continuity, and no initial conditions are needed for it.

The solution of the heat conduction equations (2.12) under the conditions (2.13) for the semi-infinite line $0 \leq \alpha < \infty$ is expressed in the form of the integrals [14]; ξ and η appear here as parameters. Then the function w_ζ can be easily determined from Equation (2.12) and the boundary condition (2.13) when $\alpha \to \infty$. Going back to the variables (2.11) we obtain

$$\mathbf{w}^*(\xi, \eta, \zeta, t) = \frac{\zeta}{2\sqrt{\pi\nu}} \int_{t_0}^{t} \frac{[\boldsymbol{\omega}(\tau) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, \tau)]}{(t-\tau)^{3/2}} \exp\frac{-\zeta^2}{4\nu(t-\tau)}\, d\tau +$$
$$+ \frac{1}{2\sqrt{\pi\nu(t-t_0)}} \int_0^\infty \left[\exp\frac{(\zeta-\zeta_1)^2}{4\nu(t_0-t)} - \exp\frac{(\zeta+\zeta_1)^2}{4\nu(t_0-t)}\right] \mathbf{w}^*(\xi, \eta, \zeta_1, t_0)\, d\zeta_1$$
$$w_\zeta(\xi, \eta, \zeta, t) = \sqrt{\nu}\, \mathrm{Div}\left[\int_\zeta^\infty \mathbf{w}^*(\xi, \eta, \zeta_1, t)\, d\zeta_1\right], \qquad \mathbf{r} = \mathbf{r}(\xi, \eta, 0)$$

For the sake of simplicity we shall assume in the future, that at the initial instant, the flow is of potential nature, and that $\mathbf{w}^* = 0$ for $t = t_0$. Then the previous formulas yield

$$\mathbf{w}^*(\xi, \eta, \zeta, t) = \frac{\zeta}{2\sqrt{\pi\nu}} \int_{t_0}^{t} \frac{\boldsymbol{\omega}(\tau) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, \tau)}{(t-\tau)^{3/2}} \exp\frac{-\zeta^2}{4\nu(t-\tau)}\, d\tau$$
$$w_\zeta(\xi, \eta, \zeta, t) = \sqrt{\nu}\, \mathrm{Div}\left[\int_\zeta^\infty \mathbf{w}^*(\xi, \eta, \zeta_1, t)\, d\zeta_1\right] =$$
$$= \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\mathrm{Div}[\boldsymbol{\omega}(\tau) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, \tau)]}{(t-\tau)^{1/2}} \exp\frac{-\zeta^2}{4\nu(t-\tau)}\, d\tau, \qquad \mathbf{r} = \mathbf{r}(\xi, \eta, 0) \tag{2.14}$$

In the domain D_Σ we shall write analogously to (2.11)

$$x_3 = \nu^{1/2}\beta, \qquad w_3 = \nu^{1/2} w_\beta \tag{2.15}$$

Then we shall change to the variables x_1, x_2, β in the equations of motion (1.7), and make $\nu \to 0$. We obtain as in (2.12) and (2.13) the equations of the boundary layer and the boundary conditions in the form

$$\frac{\partial w_i}{\partial t} = \frac{\partial^2 w_i}{\partial\beta^2}, \qquad \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_\beta}{\partial\beta} = 0 \qquad (i = 1, 2) \tag{2.16}$$
$$\frac{\partial w_i}{\partial\beta} = -2\sqrt{\nu}\frac{\partial^2\varphi^\circ}{\partial x_i \partial x_3} \quad \text{for } \beta = 0, \qquad w_i \to 0, \qquad w_\beta \to 0 \quad \text{for } \beta \to -\infty$$

(here, as in the domain D_S, we have $s = 0$).

The boundary conditions for $\beta = 0$ express the fact that the sum $\mathbf{u} = \nabla\varphi' + \mathbf{w}$ satisfies, with an accuracy up to higher order terms, the last two conditions (1.7). The solution of the heat conduction equations for w_1 and w_2 with the boundary conditions (2.16) and for zero initial conditions is expressed by quadratures [14]. Then, we can determine w_β from the equation of continuity and the boundary condition (2.16) when $\beta \to -\infty$. Returning to the original variables (2.15) we have

$$w_i = \frac{2\sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\partial^2\varphi^\circ(x_1, x_2, 0, \tau)}{\partial x_i \partial x_3} \exp\left[\frac{-x_3^2}{4\nu(t-\tau)}\right] \frac{d\tau}{(t-\tau)^{1/2}}$$

$$w_3 = -2\sqrt{2\nu} \int_{t_0}^{t} \frac{\partial^3\varphi^\circ(x_1, x_2, 0, \tau)}{\partial x_3^3} \left\{1 - \Phi\left[\frac{-x_3}{2\sqrt{\nu(t-\tau)}}\right]\right\} d\tau$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{t^2}{2}\right) dt, \qquad \Phi(\infty) = 1 \qquad (i = 1, 2) \qquad (2.17)$$

Laplace's equation for φ° is used in the integrand of w_3. Expressions (2.14) and (2.17) which are obtained, are, as expected, functions of the boundary layer type and decay exponentially outside the domains D_S and D_Σ. Thus the components of the vector $\mathbf{w}^*$ which are tangent and normal to the surfaces S and Σ have different orders of smallness in ν, whereby both these components are of an order of magnitude larger in the domain D_S, than in D_Σ (cf.(2.14) and (2.17)).

The solutions of (2.14) and (2.17) are not valid in the domain D_Γ, adjacent to the contour Γ, along which the surfaces S and Σ intersect each other. This domain which is the intersection of the domains D_S and D_Σ has a thickness of the order $\nu^{1/2}$ on the normal to the contour Γ. Thus if we consider that $\mathbf{w}$, $\mathbf{w}_t$ are bounded in D_Γ, then the derivatives can be estimated by

$$|\partial w_i / \partial x_k| \sim \nu^{-1/2}, \quad |\nu\Delta\mathbf{w}| \sim 1 \quad \text{in } D_\Gamma \quad \text{for } i, k = 1, 2, 3$$

But then, there follows from the equations of motion (1.7) in which $\mathbf{u}$ is replaced by $\mathbf{w}$, and p by s that $|\nabla s| \sim 1$ in D_Γ. Since $s = 0$ outside D_Γ, the thickness of the domain D_Γ is a quantity of the order $\nu^{1/2}$, then $s \sim \nu^{1/2}$ in D_Γ. The estimates

$$|\mathbf{w}| \sim 1, \qquad |\partial w_3 / \partial x_3| \sim \nu^{-1/2}, \qquad s \sim \nu^{1/2} \text{ in } D_\Gamma \qquad (2.18)$$

will be used in the future.

The functions $\mathbf{v}^1$, q^1, f^1 from the expressions (2.1) must compensate the disparity in the condition $\mathbf{u}\cdot\mathbf{n} = (\boldsymbol{\omega} \times \mathbf{r})\cdot\mathbf{n}$ on S, and also in the first two conditions (1.7) on Σ, which is obtained by the solution in the boundary layer of $\mathbf{w}$, s. We shall write those boundary conditions, substituting in them $\mathbf{u}$ from (2.1) and taking into account (2.3) and the boundary conditions (2.4)

$$\frac{\partial\varphi^1}{\partial n} = -\frac{\mathbf{w}\cdot\mathbf{n}}{\sqrt{\nu}} \text{ on } S, \qquad \frac{\partial\varphi^1}{\partial x_3} + \frac{w_3}{\sqrt{\nu}} = \frac{\partial f^1}{\partial t}$$

$$-\frac{\partial\varphi^1}{\partial t} - gf^1 + \frac{s}{\sqrt{\nu}} = 2\sqrt{\nu}\,\frac{\partial^2\varphi^\circ}{\partial x_3^2} + 2\sqrt{\nu}\,\frac{\partial w_3}{\partial x_3} \quad \text{on } \Sigma \qquad (2.19)$$

The functions φ^1 and f^1 satisfying the conditions (2.19) are sought, as in (2.5), in the form

$$\varphi^1 = \chi + \sum_{k=0}^{\infty} a_k^1(t)\,\psi_k(x_1, x_2, x_3), \qquad f^1 = \sum_{k=0}^{\infty} b_k^1(t)\,\psi_k(x_1, x_2, 0) \quad (2.20)$$

Here a_k^1 and b_k^1 are unknown coefficients, χ is a function harmonic in D and satisfying Neumann's conditions

$$\frac{\partial\chi}{\partial n} = -\frac{\mathbf{w}\cdot\mathbf{n}}{\sqrt{\nu}} \quad \text{on } S, \qquad \frac{\partial\chi}{\partial x_3} = -\frac{w_3}{\sqrt{\nu}} \quad \text{on } \Sigma \tag{2.21}$$

The right-hand sides of the conditions (2.21) are determined by Equations (2.14) and (2.17) in which it is necessary to make, respectively, $\zeta = 0$, and $x_3 = 0$, whereupon $\mathbf{w}\cdot\mathbf{n} = w_\zeta \sim \sqrt{\nu}$ on S, $w_3 \sim \nu$ on Σ. Neumann's problem with the conditions (2.21) can be solved since

$$\int_{S+\Sigma} \frac{\partial\chi}{\partial n}\, dS = -\frac{1}{\sqrt{\nu}} \int_{S+\Sigma} \mathbf{w}\cdot\mathbf{n}\, dS = -\frac{1}{\sqrt{\nu}} \int_D \operatorname{div}\mathbf{w}\, dV = 0$$

by virtue of the equation of continuity of $\mathbf{w}$.

Let us substitute the expansion (2.20) into the conditions (2.19). The condition (2.19) on S will be satisfied in agreement with the equations (2.7) and (2.21). In the conditions (2.19) on the surface Σ we shall express $\partial\chi/\partial x_3$, $\partial\psi_k/\partial x_3$ by equalities (2.7) and (2.21), and then we shall expand those conditions in Fourier series of the functions $\psi_k(x_1, x_2, 0)$ on Σ. We shall obtain, as in (2.9), the following equations:

$$(b_k^1)^{\cdot} = \lambda_k^2 a_k^1, \qquad (a_k^1)^{\cdot} + g b_k^1 + \frac{\partial}{\partial t}\Big(\int_\Sigma \chi\psi_k dS\Big) + \Delta_k = 0$$

$$\Delta_k = \int_\Sigma \Big(2\sqrt{\nu}\, \frac{\partial^2\varphi^\circ}{\partial x_3^2} + 2\sqrt{\nu}\, \frac{\partial w_3}{\partial x_3} - \frac{s}{\sqrt{\nu}}\Big)\psi_k dS \qquad (k = 0, 1, \ldots) \tag{2.22}$$

Let us estimate the components of Δ_k. Almost in the entire domain Σ except in a band of thickness $\sim\nu^{1/2}$, adjacent to the contour Γ, we have $s = 0$, $\partial w_3 / \partial x_3 \sim \nu^{1/2}$, as it follows from Equation (2.17). Thus the integrand of Δ_k is of the order $\nu^{1/2}$. In the proximity of the contour Γ in a small domain of order $\nu^{1/2}$, as can be seen from the estimates (2.18), this integrand is of order $O(1)$. Consequently, $\Delta_k \sim \nu^{1/2}$, and this term can be dropped in (2.22).

Let us now compute the Fourier coefficients of the function χ. Using the integral theorems of analysis, the boundary conditions (2.7) and (2.21) for the harmonic functions χ and ψ_k and Equation $\operatorname{div}\mathbf{w} = 0$, we shall obtain

$$\int_\Sigma \chi\psi_k dS = \frac{1}{\lambda_k^2}\int_\Sigma \chi\frac{\partial\psi_k}{\partial x_3}dS = -\frac{1}{\lambda_k^2}\int_{S+\Sigma}\chi\frac{\partial\psi_k}{\partial n}dS = -\frac{1}{\lambda_k^2}\int_{S+\Sigma}\frac{\partial\chi}{\partial n}\psi_k dS =$$
$$= \frac{1}{\sqrt{\nu}\lambda_k^2}\int_{S+\Sigma}\mathbf{n}\cdot\mathbf{w}\psi_k dS = -\frac{1}{\sqrt{\nu}\lambda_k^2}\int_D \operatorname{div}(\psi_k\mathbf{w})\, dV = -\frac{1}{\sqrt{\nu}\lambda_k^2}\int_D \nabla\psi_k\cdot\mathbf{w}dV \tag{2.23}$$

($\mathbf{n}$ is the unit vector of the normal interior to D).

The function $\mathbf{w}$ is bounded for $\nu \to 0$ in D_S and D_Γ, is small in D_Σ (see (2.14),(2.17) and (2.18)) and is practically equal to zero in the remaining part of the domain D.

Since the volume of the domain D_Γ is of order ν, and D_S of order

$\nu^{1/2}$, the basic contribution in the last integral (2.23) is brought by the integral over the domain D_S. In the domain D_S we can write $\mathbf{w} = \mathbf{w}^*$ with an accuracy up to higher order terms. Furthermore, since the function $\mathbf{w}^*$ decays rapidly when ζ increases, the integration over D_S can be replaced by an integration along ζ from 0 to ∞ over the surface S. Then we obtain from (2.23)

$$\int_\Sigma \chi\psi_k dS = -\frac{1}{\sqrt{\nu}\lambda_k^2}\int_{D_s} \nabla\psi_k\cdot\mathbf{w}^* dV = -\frac{1}{\sqrt{\nu}\lambda_k^2}\int_S \nabla\psi_k\cdot\left(\int_0^\infty \mathbf{w}^* d\zeta\right) dS \quad (2.24)$$

Using Formulas (2.14) and (2.5) we compute the integral

$$\int_0^\infty \mathbf{w}^* d\zeta = \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_{t_0}^t \frac{\omega\times\mathbf{r}-\nabla\varphi^\circ}{(t-\tau)^{1/2}} d\tau = \quad (2.25)$$

$$= -\frac{\sqrt{\nu}}{\sqrt{\pi}}\Big[\sum_{i=1}^3(\mathbf{r}\times\mathbf{e}_i+\nabla\Phi_i)\int_{t_0}^t \frac{\omega_i(\tau)\,d\tau}{(t-\tau)^{1/2}} + \sum_{j=1}^\infty \nabla\psi_j\cdot\int_{t_0}^t \frac{a_j^\circ(\tau)\,d\tau}{(t-\tau)^{1/2}}\Big]$$

In the last summation the term with $j = 0$ has been omitted, since $\psi^\circ = \mathrm{const}$, $\nabla\psi^\circ = 0$. Substituting (2.25) into (2.24) and then substituting (2.24) into Equation (2.22)

$$(b_k^1)^\cdot = \lambda_k^2 a_k^1 \quad (k = 1, 2, \ldots) \quad (2.26)$$

$$(a_k^1)^\cdot + g b_k^1 + \frac{1}{\sqrt{\pi}\lambda_k^2}\frac{d}{dt}\Big[\sum_{i=1}^3 C_{ik}\int_{t_0}^t \frac{\omega_i(\tau)\,d\tau}{\sqrt{t-\tau}} + \sum_{j=1}^\infty D_{jk}\int_{t_0}^t \frac{a_j^\circ(\tau)\,d\tau}{\sqrt{t-\tau}}\Big] = 0$$

Here the constants C_{ik} and D_{jk} are determined by Equations (2.27)

$$C_{ik} = \int_S (\mathbf{r}\times\mathbf{e}_i+\nabla\Phi_i)\cdot\nabla\psi_k dS, \quad D_{jk} = \int_S \nabla\psi_j\cdot\nabla\psi_k dS \quad (i = 1, 2, 3;\ j, k = 1, 2, \ldots)$$

Let us investigate separately the case $k = 0$. It corresponds to $\lambda_0 = 0$ and, as follows from (2.7) and the condition (2.8), $\psi^\circ = \mathrm{const} = 1/\sqrt{\Sigma}$ where Σ is the area of the nondisturbed free surface Σ. As a consequence of the invariance of the volume of the liquid, we have

$$\int_\Sigma f\,dS = \int_\Sigma f^\circ\,dS + \nu^{1/2}\int_\Sigma f^1\,dS = 0$$

Substituting into this equation the expansions (2.5) and (2.20) we obtain $b_0^\circ = b_0^1 = 0$. Then, from Equations (2.9) and (2.22) we obtain for $k = 0$

$$a_0(t) = a_0^\circ + \nu^{1/2} a_0^1 = A_{10}\int_{t_0}^t g_1\,dt + A_{20}\int_{t_0}^t g_2\,dt - \sum_{i=1}^3 \omega_i(t)B_{i0} - \frac{\sqrt{\nu}}{\sqrt{\Sigma}}\int_\Sigma \chi\,dS + \mathrm{const} \quad (2.28)$$

If Equations (2.1),(2.3),(2.5) and (2.20) are taken into consideration, the equation of the hydrodynamic problem in the given approximation takes the form (2.29)

$$\mathbf{u} = \nabla\varphi + \mathbf{w}, \qquad p = -\varphi_t + s, \qquad \varphi = \varphi^\circ + \nu^{1/2}\varphi^1 = \sum_{i=1}^3 \omega_i\Phi_i +$$

$$+\sum_{k=1}^\infty a_k\psi_k + \nu^{1/2}\chi + c(t), \qquad a_k = a_k^\circ + \nu^{1/2}a_k^1, \qquad c = a_0\psi^\circ = \frac{a_0}{\sqrt{\Sigma}}$$

$$f = f^\circ + \nu^{1/2} f^1 = \sum_{k=1}^{\infty} b_k(t)\, \psi_k(x_1, x_2, 0), \qquad b_k = b_k^\circ + \nu^{1/2} b_k^1$$

Here w is determined by Equations (2.14) and (2.17) in the domains D_S and D_Σ and is equal to zero in the remaining part of the domain D, $s \equiv 0$ everywhere, except in the domain D_Γ, where the estimates (2.18) are valid.

The functions Φ_i, ψ_k and χ, harmonic in D, are determined by the boundary conditions (2.6),(2.7) and (2.21). The function a_0 is specified by Equation (2.28). For the summation coefficients a_k and b_k, determined by Equations (2.29) we obtain from (2.9) and (2.26), equations which are valid with an accuracy up to small terms of higher order

$$b_k^{\cdot} = \lambda_k^2 a_k, \qquad a_k^{\cdot} + g b_k - g_1 A_{1k} - g_2 A_{2k} + \sum_{i=1}^{3} \omega_i^{\cdot} B_{ik} +$$
$$+ \frac{\sqrt{\nu}}{\sqrt{\pi}\,\lambda_k^2} \frac{d}{dt}\Big[\sum_{i=1}^{3} C_{ik} \int_{t_0}^{t} \frac{\omega_i(\tau)\, d\tau}{\sqrt{t-\tau}} + \sum_{j=1}^{\infty} D_{jk} \int_{t_0}^{t} \frac{a_j(\tau)\, d\tau}{\sqrt{t-\tau}}\Big] = 0 \qquad (k = 1, 2, \ldots) \tag{2.30}$$

The obtained linear system of integro-differential equations describes the motion of the liquid. If the motion of the body is specified, i.e. the quantities g, g_1, g_2 and ω_i are known as functions of time, then the investigation of the motion of the liquid reduces to the solution of Cauchy's problem for the system (2.30) with the initial conditions

$$a_k(t_0) = \int_D \varphi(x_1, x_2, x_3, t_0)\, \psi_k(x_1, x_2, x_3)\, dV$$

$$b_k(t_0) = \int_\Sigma f(x_1, x_2, t_0)\, \psi_k(x_1, x_2, 0)\, dS \qquad (k = 1, 2, \ldots)$$

The functions φ and f must be known at the initial instant.

Instead of the potential φ for $t = t_0$, we can specify, for instance $\partial f/\partial t$, and then, in agreement with (2.30), determine

$$a_k(t_0) = \frac{b_k^{\cdot}(t_0)}{\lambda_k^2} = \frac{1}{\lambda_k^2} \int_\Sigma \frac{\partial f(x_1, x_2, t_0)}{\partial t}\, \psi_k(x_1, x_2, 0)\, dS$$

3. Equations of motion of the body with liquid. In order to describe the motion of the body with liquid, we shall simplify Equations (1.5). We shall substitute in them the solution (2.29) and compute the integrals using the notations (2.10)

$$\mathbf{Q} = m\boldsymbol{\omega} \times \mathbf{r}_c + \rho \sum_{k=1}^{\infty} b_k^{\cdot} (\mathbf{e}_1 A_{1k} + \mathbf{e}_2 A_{2k})$$
$$\mathbf{Q}^{\cdot} \equiv m\boldsymbol{\omega}^{\cdot} \times \mathbf{r}_c + \rho \sum_{k=1}^{\infty} b_k^{\cdot\cdot} (\mathbf{e}_1 A_{1k} + \mathbf{e}_2 A_{2k}) = \mathbf{F} + m\mathbf{g}$$
$$\mathbf{g} = \mathbf{g}_0 - \mathbf{R}_0^{\cdot\cdot} = -g\mathbf{e}_3 + g_1\mathbf{e}_1 + g_2\mathbf{e}_2 \qquad (m = m_1 + m_2) \tag{3.1}$$
$$\mathbf{K}^{\cdot} = \mathbf{M} + m\mathbf{r}_c \times \mathbf{g} + \rho g \sum_{k=1}^{\infty} b_k (\mathbf{e}_2 A_{1k} - \mathbf{e}_1 A_{2k})$$

The terms with an order of smallness larger than the first have been dropped. Note that the coefficients a_k and b_k as well as φ, f, ω, g_1 and g_2 and their derivatives are small quantities of the first order, the unit vectors $\mathbf{e}_i$ satisfy Equations (1.3).

In the last equation of (1.5) we shall substitute $\mathbf{u}$ from (2.29)

$$\mathbf{K} = \mathbf{J}_1 \cdot \boldsymbol{\omega} + \rho \int \mathbf{r} \times \nabla\varphi \, dV + \rho \int_D \mathbf{r} \times \mathbf{w} \, dV \tag{3.2}$$

Let us transform the first integral of (3.2) using the relations $\mathbf{r} \times \nabla\varphi = -\operatorname{rot}(\mathbf{r}\varphi)$ and (2.6), and also the theorems of Stokes and Green

$$\int_D \mathbf{r} \times \nabla\varphi dV = -\int_D \operatorname{rot}(\mathbf{r}\varphi)\, dV = -\int_D \mathbf{r} \times \mathbf{n}\varphi dS = -\int_{S+\Sigma} \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial\Phi_i}{\partial n} \varphi dS =$$

$$= -\sum_{i=1}^{3} \mathbf{e}_i \int_{S+\Sigma} \Phi_i \frac{\partial\varphi}{\partial n} dS = \sum_{i=1}^{3} \mathbf{e}_i \left(\int_\Sigma \Phi_i \frac{\partial\varphi}{\partial x_3} dS - \int_S \Phi_i \frac{\partial\varphi}{\partial n} dS \right). \tag{3.3}$$

Substituting into Equation (3.3) the value of φ given by (2.29), and using the boundary conditions (2.6),(2.7) and (2.21) for the functions Φ_i, ψ_k and χ we obtain

$$\int_D \mathbf{r} \times \nabla\varphi dV = \sum_{i=1}^{3} \mathbf{e}_i \Big(\sum_{j=1}^{3} \omega_j J_{ij} + \sum_{k=1}^{\infty} \lambda_k^2 B_{ik} a_k + \int_{S+\Sigma} \Phi_i \frac{\mathbf{w}\cdot\mathbf{n}}{\sqrt{\nu}} dS \Big)$$

$$J_{ij} = J_{ji} = -\int_{S+\Sigma} \Phi_i \frac{\partial\Phi_j}{\partial n} dS = \int_\Sigma \Phi_i \frac{\partial\Phi_j}{\partial x_3} dS - \int_S \Phi_i \frac{\partial\Phi_j}{\partial n} dS \tag{3.4}$$

Here the constants J_{ij} are the components of the tensor $\mathbf{J}_2$ of the coupled masses for the domain D, in which the free surface Σ has been replaced by a plane rigid wall. The constants B_{ik} are determined by the Equations (2.10).

Let us transform, as in (2.23), the integral of Formula (3.4)

$$\int_{S+\Sigma} \Phi_i \mathbf{w}\cdot\mathbf{n} dS = -\int_D \operatorname{div}(\Phi_i \mathbf{w})\, dV = -\int_D \nabla\Phi_i \cdot \mathbf{w} dV \tag{3.5}$$

Substituting (3.5) into (3.4) and then (3.4) into Formula (3.2) to obtain

$$\mathbf{K} = \mathbf{J}\cdot\boldsymbol{\omega} + \rho \sum_{k=1}^{\infty} \lambda_k^2 a_k \Big(\sum_{i=1}^{3} B_{ik} \mathbf{e}_i \Big) + \rho \int_D \Big[\mathbf{r} \times \mathbf{w} - \sum_{i=1}^{3} \mathbf{e}_i (\nabla\Phi_i \cdot \mathbf{w}) \Big] dV$$

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \tag{3.6}$$

Using the identity

$$\mathbf{r} \times \mathbf{w} - \sum_{i=1}^{3} \mathbf{e}_i (\nabla\Phi_i \cdot \mathbf{w}) = -\sum_{i=1}^{3} \mathbf{e}_i [(\mathbf{r} \times \mathbf{e}_i + \nabla\Phi_i)\cdot\mathbf{w}]$$

and transforming the integral, which enters Equation (3.6), as in (2.24), we obtain

$$\int_D \Big[\mathbf{r} \times \mathbf{w} - \sum_{i=1}^{3} \mathbf{e}_i (\nabla\Phi_i \cdot \mathbf{w}) \Big] dV = -\sum_{i=1}^{3} \mathbf{e}_i \int_S (\mathbf{r} \times \mathbf{e}_i + \nabla\Phi_i) \Big(\int_0^\infty \mathbf{w}^* d\zeta \Big) dS \tag{3.7}$$

Taking into consideration Equation (3.7) and Formula (2.25), Expression (3.6) is brought to the final form

$$\mathbf{K} = \mathbf{J}\cdot\boldsymbol{\omega} + \rho \sum_{i=1}^{3} \mathbf{e}_i \left\{ \sum_{k=1}^{\infty} B_{ik}\lambda_k{}^2 a_k + \frac{\sqrt{\nu}}{\sqrt{\pi}} \left[\sum_{j=1}^{3} E_{ij} \times \right.\right.$$
$$\left.\left. \times \int_{t_0}^{t} \frac{\omega_j(\tau)\,d\tau}{\sqrt{t-\tau}} + \sum_{k=1}^{\infty} C_{ik} \int_{t_0}^{t} \frac{a_k(\tau)\,d\tau}{\sqrt{t-\tau}} \right] \right\} \qquad (\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2) \qquad (3.8)$$

$$E_{ij} = \int_S (\mathbf{r}\times\mathbf{e}_i + \nabla\Phi_i)\cdot(\mathbf{r}\times\mathbf{e}_j + \nabla\Phi_j)\,dS \qquad (i, j = 1, 2, 3) \qquad (3.9)$$

The constants C_{ik} are determined by Formulas (2.27), and the constants E_{ij} differ only by constant factors from the elements of the tensor $\mathbf{B}$, introduced in [9].

The dynamics of the body with a liquid is described by the integro-differential equations (2.30) and (3.1) in which $\mathbf{K}$ must be substituted from (3.8), and also by the kinematic relations (1.3). Then, the motion of the body is characterized by the parameters $\mathbf{R}_0$, $\mathbf{R}_0'$, $\mathbf{e}_i$, $\boldsymbol{\omega}_i$, and the motion of the liquid by the quantities a_k and b_k. In fact those parameters should be specified as initial conditions. The function $a_0(t)$, determined by Equation (2.28) does not affect the motion of the body and the distribution of the velocities in the liquid, and hence is not essential.

In the case of systems more complex than one solid with liquid, it is also necessary to add some complementing equations.

If the body has several liquid-filled cavities, there corresponds a system (2.30) to each cavity, and similar groups of terms corresponding to each cavity are summed in Formulas (3.1) and (3.8).

In Equations (2.30),(3.1) and (3.8) one finds the constants λ_k and J_{ij} (through the tensor $\mathbf{J}_2$), A_{ik}, B_{ik}, C_{ik}, D_{jk} and E_{ij}, which depend only on the form of the cavity and the level of the liquid. In order to determine them, it is necessary to solve the three boundary problems (2.6) and the eigenvalue problem (2.7) once for a given form of cavity (it is not necessary to solve the boundary value problem (2.21) for the functions χ), and then calculate the integrals (2.10),(2.27) and (3.4) for J_{ij}, (3.9). The volume of calculations though, will be almost the same as in the case of the ideal liquid for which it is not necessary to find the coefficients C_{ik} and E_{ij}.

Cauchy's problem for Equations (2.30),(3.1),(3.7) and (1.3) can be solved either by means of a direct numerical integration, or by means of different approximate methods. Note that these equations can be significantly simplified. First, in practical problems, it is sufficient to limit oneself to a small number of Fourier coefficients a_k, b_k, i.e. consider only a few forms (or even one only) of the principal oscillations of the liquid. Then it is simply assumed that the other coefficients are equal to zero, and the equations of the system (2.30) which corresponds to them are discarded.

Secondly, Equations (2.30) and (3.8) include the parameter $\sqrt{\nu} \ll 1$ in the integral terms, which can serve as a basis to justify the use of the small parameters method. Thirdly, in many cases the mass of liquid is small with respect to the mass of the body, thus the terms related to the motion of the liquid can be considered as perturbing. Formally, this leads to the fact that ρ can be considered as a small parameter in Equations (3.1) and (3.8). The simplifications indicated can be widely used in practical problems.

Equations (2.30),(3.1) and (3.8) are derived in the linear approximation with respect to the amplitude of the oscillations (the quantities a_k, b_k, $\boldsymbol{\omega}$, θ_1 and θ_2 and their derivatives and integrals are small quantities of the first order) and with an error of the order of ν in the small parameter

$\nu \ll 1$. If the approximations made are to be valid over the entire interval of motion, it is necessary that the integrals

$$\int_{t_0}^{t} \frac{\omega_i(\tau)\,d\tau}{\sqrt{t-\tau}}, \qquad \int_{t_0}^{t} \frac{a_k(\tau)\,d\tau}{\sqrt{t-\tau}}$$

entering Formulas (2.30) and (3.8) remain small quantities of the first order in that interval. This is possible, either if the interval of motion is sufficiently small (of the order of the unity, i.e. of the characteristic period of oscillations), or if the functions ω_i and a_k oscillate around zero. Otherwise the boundary layer grows with time, and the flow of liquid in the cavity will never be close to a potential one. The vortex motion of the liquid in the cavity of the solid, even in the absence of viscosity, has quite a complicated nature (see [15]).

4. Forced oscillations of the liquid. Let the motion of the body be given by Formulas

$$g_i = g_{i0}e^{\mu t} \quad (i = 1, 2), \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_0 e^{\mu t} = \sum_{i=1}^{3} \mathbf{e}_i \omega_{i0} e^{\mu t}, \qquad g = \text{const} \tag{4.1}$$

where μ is a complex number and g_{i0} and ω_{i0} are constants. Furthermore, let us assume that the unit vectors $\mathbf{e}_i$ of the system of coordinates associated with the body, appear in the form $\mathbf{e}_i = \mathbf{e}_{i0} + \delta\mathbf{e}_i$. Here the $\mathbf{e}_{i0}$ are the unit vectors of the cartesian system of coordinates $Oy_1y_2y_3$ (see Section 1), and $\delta\mathbf{e}_i$ are small quantities of the first order, proportional to $e^{\mu t}$. Taking the relations (1.3) and (4.1) into account, the unit vectors $\mathbf{e}_i$ can be expressed (in the linear approximation) in the form

$$\mathbf{e}_i = \mathbf{e}_{i0} + (\boldsymbol{\omega} \times \mathbf{e}_{i0})/\mu = \mathbf{e}_{i0} + (\boldsymbol{\omega}_0 \times \mathbf{e}_{i0})\,e^{\mu t}/\mu \tag{4.2}$$

Let us determine the motion of the liquid for which the functions $\mathbf{u}$, p and f depend on time by means of the multiplying factor $e^{\mu t}$. The process of solution will not differ much from the investigations of the Sections 2 and 3 in which Cauchy's problem was studied. The solution of the problem (1.7) is again sought in the form (2.1), where all the terms of the expansions are proportional to $e^{\mu t}$. The relations (2.3) are obtained again for the functions $\mathbf{v}^k$ and q^k ; for φ° and f° we get the relations (2.5), in which the coefficients a_k° and b_k° are proportional to $e^{\mu t}$.

As before, the function $\mathbf{w}$ satisfies the equations and boundary conditions (2.12) and (2.13) in the domain D_S , and (2.16) in the domain D_Σ. The solution of those boundary value problems, which depend on time as $e^{\mu t}$, has the form (see [9 and 10])

$$\begin{aligned}
&\mathbf{w}^*(\xi, \eta, \zeta, t) = [\boldsymbol{\omega}(t) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, t)] \exp\left(\sqrt{\mu/\nu}\,\zeta\right) \qquad (\mathbf{r} = \mathbf{r}(\xi, \eta, 0))\\
&w_\zeta(\xi, \eta, \zeta, t) = \sqrt{\nu/\mu}\,\mathrm{Div}\,[\nabla\varphi^\circ(\mathbf{r}, t) - \boldsymbol{\omega}(t) \times \mathbf{r}] \exp\left(\sqrt{\mu/\nu}\,\zeta\right)\\
&w_i(x_1, x_2, x_3, t) = 2\frac{\sqrt{\nu}}{\sqrt{\mu}} \frac{\partial^2\varphi^\circ(x_1, x_2, 0, t)}{\partial x_i \partial x_3} \exp\left(-\frac{\sqrt{\mu}\,x_3}{\sqrt{\nu}}\right) \qquad (i = 1, 2)\\
&w_3(x_1, x_2, x_3, t) = -\frac{2\nu}{\mu} \frac{\partial^3\varphi^\circ(x_1, x_2, 0, t)}{\partial x_3^3} \exp\left(-\frac{\sqrt{\mu}\,x_3}{\sqrt{\nu}}\right)
\end{aligned} \tag{4.3}$$

Here and further on $\sqrt{\mu}$ represents the root for which $\mathrm{Re}\sqrt{\mu} < 0$. The functions $\boldsymbol{\omega}$, φ° and $\mathbf{w}$ are proportional to $e^{\mu t}$.

The functions φ^1 and f^1 satisfy the relations (2.20), in which the function χ satisfying the boundary conditions (2.21) and the coefficients a_k^1 and b_k^1 are proportional to $e^{\mu t}$. Instead of Equations (2.25) we obtain by means of (4.3),(2.5):

$$\int_0^\infty \mathbf{w}^*\,d\zeta = -\frac{\sqrt{\nu}}{\sqrt{\mu}}(\boldsymbol{\omega} \times \mathbf{r} - \nabla\varphi^\circ) = \frac{\sqrt{\nu}}{\sqrt{\mu}}\Big[\sum_{i=1}^{3}(\mathbf{r} \times \mathbf{e}_i + \nabla\Phi_i)\,\omega_i + \sum_{j=1}^{\infty} \nabla\varphi_j a_j^\circ\Big] \tag{4.4}$$

Substituting (4.4) into (2.24) and taking (2.27) into consideration

$$\int_{\Sigma} \chi \psi_k \, dS = -\frac{1}{\lambda_k^2 \sqrt{\mu}} \left(\sum_{i=1}^{3} C_{ik} \omega_i + \sum_{j=1}^{\infty} D_{jk} a_j^{\circ} \right) \tag{4.5}$$

The motion of the liquid in the considered approximation can be described as before by the relations (2.29). In Equations (2.29) the function w is now determined by Formulas (4.3) in the domains D_S and D_Σ, and $w = 0$ in the remaining part of the domain D. The function s differs from zero only in the domain D_Γ, where the estimates (2.18) are valid. The coefficients a_k and b_k which enter (2.29) have now the form

$$a_k = a_k^{\circ} + \nu^{1/2} a_k^{1} = c_k e^{\mu t}, \qquad b_k = b_k^{\circ} + \nu^{1/2} b_k^{1} = d_k e^{\mu t}, \qquad k = 1, 2, \ldots \tag{4.6}$$

(where c_k and d_k are constants).

From Equations (2.9) and (2.22) and taking the equalities (4.1),(4.5) and (4.6) into consideration, and $\Delta_k = 0$, we obtain algebraic equations for the constants c_k and d_k

$$\mu d_k = \lambda_k^2 c_k \qquad (k = 1, 2, \ldots) \tag{4.7}$$

$$\mu c_k + g d_k - g_{10} A_{1k} - g_{20} A_{2k} + \mu \sum_{i=1}^{3} \omega_{i0} B_{ik} - \frac{\sqrt{\nu\mu}}{\lambda_k^2} \left(\sum_{i=1}^{3} C_{ik} \omega_{i0} + \sum_{j=1}^{\infty} D_{jk} c_j \right) = 0$$

The case $k = 0$ is considered separately. As in the relation (2.28) we obtain

$$a_0(t) = e^{\mu t} \left(\frac{g_{10} A_{10} + g_{20} A_{20}}{\mu} - \sum_{i=1}^{3} B_{i0} \omega_{i0} \right) - \frac{\sqrt{\nu}}{\sqrt{\Sigma}} \int_{\Sigma} \chi \, dS$$

The function $a_0(t)$, as can be seen from (2.29), acts only on the distribution of the pressures in the liquid; the velocity of the liquid, and the motion of the body are independent from it.

Let us compute the kinetic moment $\mathbf{K}$, which proceeds from Formula (3.6). Substituting in sequence the relations (3.7),(4.4),(4.1) and (4.6) into (3.6) we obtain as in (3.8)

$$\mathbf{K} = e^{\mu t} \left\{ \mathbf{J} \cdot \boldsymbol{\omega}_0 + \rho \sum_{i=1}^{3} \mathbf{e}_i \left[\sum_{k=1}^{\infty} B_{ik} \lambda_k^2 c_k - \frac{\sqrt{\nu}}{\sqrt{\mu}} \left(\sum_{j=1}^{3} E_{ij} \omega_{j0} + \sum_{k=1}^{\infty} C_{ik} c_k \right) \right] \right\} \tag{4.8}$$

Here the small terms of higher order have been neglected and the notations (2.27) and (3.9) have been used. Let us write now, taking the equalities (4.1),(4.6) and (4.8) into consideration, the equations of motion of the body (3.1) in the case of forced oscillations

$$\mathbf{Q} = e^{\mu t} \left[m \boldsymbol{\omega}_0 \times \mathbf{r}_c + \rho\mu \sum_{k=1}^{\infty} d_k (\mathbf{e}_1 A_{1k} + \mathbf{e}_2 A_{2k}) \right]$$

$$\mu \mathbf{Q} = \mathbf{F} - mg\mathbf{e}_3 + m(g_{10}\mathbf{e}_1 + g_{20}\mathbf{e}_2) e^{\mu t} \tag{4.9}$$

$$\mu \mathbf{K} = \mathbf{M} - m\mathbf{r}_c \times \mathbf{e}_3 g + m e^{\mu t} \mathbf{r}_c \times (g_1 \mathbf{e}_1 + g_2 \mathbf{e}_2) + \rho g e^{\mu t} \sum_{k=1}^{\infty} d_k (\mathbf{e}_2 A_{1k} - \mathbf{e}_1 A_{2k})$$

Thus, in the case of forced oscillations, the motion of the body with liquid is described by Equations (4.1),(4.2) and (4.7) to (4.9). Taking μ, g_{i0} and ω_{i0} as constants, and solving the system of linear equations (4.7), we can find the coefficients c_k and d_k which describe the motion of the liquid. Then, Equations (4.9) and (4.8) determine the force $\mathbf{F}$ and the moment $\mathbf{M}$, necessary to maintain such a motion of the body with liquid. Furthermore, using Equations (4.7) to (4.9), it is not difficult to find the forced oscillations of the body with liquid subject to the influence of forces and moments changing according to a law in $e^{\mu t}$ (in particular, when μ is a purely imaginary quantity and the law is sinusoidal). Finally, Equations (4.7) to (4.9) can be used to determine the proper oscillations of the body with liquid. All those problems are purely algebraic, and to simplify them one can use the considerations mentioned at the end of Section 3.

Let us also point out that Equations (4.4) and (4.7) to (4.9) can be immediately obtained from the relations (2.25),(2.30),(3.8) and (3.1), respectively, by making the following substitutions:

$$\frac{dF}{dt} \to \mu F, \qquad \int_{t_0}^{t} \frac{F(\tau)\, d\tau}{\sqrt{t-\tau}} \to -\frac{\sqrt{\pi}}{\sqrt{\mu}} F \tag{4.10}$$

in the corresponding equations of Sections 2 and 3.

Here F is a function of the form $\mathrm{const}\cdot e^{\mu t}$, entering those relations. The first equality (4.10) is obvious. To justify the second let us make the change of variables $\tau = t - x^2$ in the integrand and let us take the limit when $t_0 \to -\infty$

$$\lim_{t_0 \to -\infty} \int_{t_0}^{t} \frac{e^{\mu\tau}\, d\tau}{\sqrt{t-\tau}} = 2e^{\mu t} \lim_{t_0 \to -\infty} \int_{0}^{\sqrt{t-t_0}} \exp(-\mu x^2)\, dx = -\frac{\sqrt{\pi}}{\sqrt{\mu}} e^{\mu t} \quad (\operatorname{Re} \sqrt{\mu} < 0)$$

Here the integral is computed by means of a known formula [16]. Although it converges only for $\operatorname{Re}\mu > 0$, nevertheless the formal substitution (4.10) in the relations of Sections 2 and 3 yields for any μ the exact Expressions (4.4),(4.5),(4.7) to (4.9) derived earlier in the case of forced oscillations.

5. Particular cases. (1). If we make $\nu = 0$ in the relations of the Sections 1 to 4, we obtain the description of the motion of a body, partially filled with an ideal liquid [1 to 6]. In particular, for $\nu = 0$ Equations (2.30),(3.8) together with (1.3) and (3.1) become (with the same notations) the general equations of the motion of a body with an ideal liquid [3].

2). Let the liquid entirely fill the cavity; there is no free surface Σ any more. Then the problem (2.7) has trivial solutions only, and we can take $\psi_k \equiv 0$ for $k = 0, 1, \ldots$.

As before, the solution of the hydrodynamics problem is represented by (2.29), but since all the $\psi_k = 0$, the coefficients a_k and b_k and the function f are inessential. Equations (2.30) and (4.7) should be neglected, and in the relations (3.1),(3.8),(4.8),(4.9) the terms including the coefficients a_k, b_k, c_k, d_k, should be dropped, i.e. simply assume $a_k = b_k = c_k = d_k = 0$ for $k = 0, 1, \ldots$. Then the formulas for the kinetic moment (3.8) and (4.8) and the other relations will be (after changing the notations) in agreement with the corresponding formulas of [9], in which the motion of a body with a cavity completely filled with a low viscosity liquid has been considered.

3). We shall also investigate the proper oscillations of a viscous liquid in a container at rest. The coefficients a_k and b_k are sought in the form (4.6). Assuming in (4.7) $\sigma_{10} = \sigma_{20} = \omega_{i0} = 0$ for $i = 1, 2, 3$ and eliminating d_k, brings Equations (4.7) to the form

$$(\mu^2 + \lambda_k^2 g)\, c_k = \frac{\sqrt{\nu\mu}\, \mu}{\lambda_k^2} \sum_{j=1}^{3} D_{jk} c_j \qquad (k = 1, 2, \ldots) \tag{5.1}$$

The eigen numbers μ of the problem of the free oscillations of a viscous liquid in a container are determined from the conditions of existence of a nonzero solution c_k of the linear homogeneous system (5.1). For $\nu = 0$, (5.1) yields the eigen numbers of the problem of the oscillations of an ideal liquid $\mu_n = \pm i\lambda_n \sqrt{g}$. Here $\lambda_n \sqrt{g}$ are the frequencies of the free oscillations assumed not to be multiples of one another: $\lambda_n \neq \lambda_m$ for $n \neq m$, $n, m = 1, 2, \ldots$ For $\mu = \mu_n$, obviously, we have $c_k = 0$ for $k \neq n$.

To determine the eigen numbers, and the form of the oscillations of the viscous liquid for $\nu \ll 1$, we shall use the perturbation method. We shall find the natural oscillation close to the nth oscillation of the ideal liquid, assuming

$$\mu = \mu_n + \sqrt{\nu}\delta, \quad \mu_n = \pm i\lambda_n \sqrt{g}, \quad c_k / c_n = O(\sqrt{\nu}) \qquad (k \neq n) \tag{5.2}$$

Then the system (5.1) has the form

$$2\delta\mu_n c_n = \sqrt{\nu\mu_n}\mu_n\lambda_n^{-2} D_{nn} c_n, \qquad (\lambda_k^2 - \lambda_n^2)\, g c_k = \sqrt{\nu\mu_n}\mu_n\lambda_k^{-2} D_{nk} c_n \tag{5.3}$$

with an accuracy up to terms of higher order ($k \neq n$).

From the relations (5.3) we determine easily δ and c_k for $k \neq n$. Using the notations (5.2), the condition $\operatorname{Re}\sqrt{\mu_n} < 0$, and also the first relations (4.7), we finally obtain

$$\mu = \pm i\lambda_n \sqrt{g} - \frac{(1 \pm i)\sqrt{\nu}\, g^{1/4} D_{nn}}{2\sqrt{2}\,\lambda_n^{3/2}}, \qquad d_n = \frac{\lambda_n^2 c_n}{\mu}$$

$$c_k = \frac{(1 \mp i)\sqrt{\nu}\,\lambda_n^{3/2} D_{nk} c_n}{\sqrt{2}\, g^{1/4} \lambda_k^2 (\lambda_n^2 - \lambda_k^2)}, \qquad d_k = \mp \frac{i\lambda_k^2 c_k}{\lambda_n \sqrt{g}} \qquad (k \neq n) \qquad (5.4)$$

The coefficient c_n remains arbitrary and determines the amplitude of the oscillation. Since $D_{nn} > 0$ (see (2.27)), then $\operatorname{Re} \mu < 0$ and the viscosity leads to a damping of the oscillations. Furthermore, from the relation (5.4) for μ, it can be seen that the viscosity yields also a decrease in the frequency of the oscillations which is equal to the decrement in damping. The natural oscillations of the liquid are determined by Formulas (2.29) in which we substitute (4.6) and (5.4). Taking (2.27) and (2.8) into consideration, it is found that the first formula of (5.4) coincides (with the same notations) with the equation in [10] giving the eigen numbers of the problem of the free oscillations of a viscous liquid. In the paper [10] computations are made for a few specific forms of the cavity. Free and forced oscillations of a body with liquid are investigated analogously.

BIBLIOGRAPHY

1. Moiseev, N.N. and Rumiantsev, V.V., Dinamika tela s polostiami, soderzhashchimi zhidkost' (Dynamics of Bodies with Cavities Filled with Liquids). M., Izd.Nauka, 1965.
2. Sretenskii, I.N., Kolebanie zhidkosti v podvizhnom sosude (Oscillations of a liquid in a moving container). Izv.Akad.Nauk SSSR, OTN, № 10, 1951.
3. Moiseev, N.N., Zadacha o dvizhenii tverdogo tela, soderzhashchego zhidkie massy, imeiushchie svobodnuiu poverkhnost' (Problem of the motion of a rigid body, including liquid masses having a free surface). Mat.sbor., Vol.32, № 1, 1953.
4. Okhotsimskii, D.E., K teorii dvizheniia tela s polostiami, chastichno zapolnennymi zhidkost'iu (On the theory of the motion of a body with cavities partly filled with liquids). PMM Vol.20, № 1, 1956.
5. Narimanov, G.S., O dvizhenii tverdogo tela, polost' kotorogo chastichno zapolnena zhidkost'iu (On the motion of a solid, the cavity of which is partly filled with a liquid). PMM Vol.20, № 1, 1956.
6. Rabinovich, B.I., Ob uravneniiakh vozmushchennogo dvizheniia tverdogo tela s tsilindricheskoi polost'iu, chastichno zapolnennoi zhidkost'iu (On the equations of the perturbed motion of a solid with a cylindrical cavity partly filled with a liquid). PMM Vol.20, № 1, 1956.
7. Krasnoshchekov, P.S., O kolebaniiakh fizicheskogo maiatnika, imeiushchego polosti, zapolnennye viazkoi zhidkost'iu (On oscillations of a physical pendulum having cavities filled with a viscous liquid). PMM Vol.27, №2, 1963.
8. Krushinskaia, S.I., Kolebaniia tiazheloi viazkoi zhidkosti v podvizhnom sosude (Oscillations of a heavy viscous liquid in a moving container). Zh.vychisl.Mat.mat.Fiz., Vol.5, № 3, 1965.
9. Chernous'ko, F.L., Dvizhenie tela s polost'iu, zapolnennoi viazkoi zhidkost'iu, pri bol'shikh chislakh Reinol'dsa (Motion of a body with a cavity filled with a viscous fluid at large Reynolds numbers). PMM Vol. 30, № 3, 1966.
10. Chernous'ko, F.L., O svobodnykh kolebaniiakh viazkoi zhidkosti v sosude (On the free oscillations of a viscous liquid in a container). PMM Vol.30, № 5, 1966.
11. Variatsionnye metody v zadachakh o kolebanii zhidkosti i tela s zhidkost'iu (Variational Methods in Problems of Oscillations of Liquids and Bodies with Liquid). Collected Papers, M. Izd.Akad.Nauk SSSR, 1962.
12. Moiseev, N.N. and Petrov, A.A., Chislennye metody rascheta sobstvennykh chastot kolebanii ogranichennogo ob"ema zhidkosti (Numerical Method of Calculation of the Natural Frequencies of Oscillation of a Bounded Volume of Liquid). M.Izd.Akad.Nauk SSSR, 1966.
13. Vishik, M.I. and Liusternik, L.A., Reguliarnoe vyrozhdenie i pogranichnyi sloi dlia lineinykh differentsial'nykh uravnenii s malym parametrom (Regular degeneration and boundary layer for linear differential equations with a small parameter). Usp.mat.Nauk, Vol.12, № 5, 1957.

14. Smirnov, V.I., Kurs vysshei matematiki (Course on Higher Mathematics). Vols. 2 and 4, M., Fizmatgiz, 1958.
15. Sobolev, S.L., O dvizhenii simmetrichnogo volchka s polost'iu napolnennoi zhidkost'iu (On the motion of a symmetrical top with a cavity filled with a liquid). PMTF, № 3, 1960.
16. Gradshtein, I.S. and Ryzhik, I.M., Tablitsy integralov, summ, riadov i proizvedenii (Tables of Integrals, Sums, Series and Transformations). 4th edition, M., Fizmatgiz, 1962.

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